

## ON THE NEGATIVE PELL EQUATION

$$y^2 = 3x^2 - 2$$

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**ABSTRACT:**

The binary quadratic equation represented by the negative Pellian  $y^2 = 3x^2 - 2$  is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola and special Pythagorean triangle.

**KEYWORDS:** Binary quadratic, hyperbola, parabola, integral solutions, Pell equations.

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**INTRODUCTION:**

Diophantine equation of the form  $y^2 = Dx^2 + 1$ , where  $D$  is a given positive square-free integer is known as Pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive Pell equation  $y^2 = Dx^2 + 1$  has infinitely many distinct integer solutions where as the negative Pell equation  $y^2 = Dx^2 - 1$  does not always have a solution. In [1], an elementary proof of a criterium for the solvability of the Pell equation  $X^2 - Dy^2 = -1$  where  $D$  is any positive non-square integer has been presented. For examples the equations  $y^2 = 3x^2 - 1$ ,  $y^2 = 7x^2 - 4$  have no integer solution whereas  $y^2 = 65x^2 - 1$ ,  $y^2 = 202x^2 - 1$  have integer

solutions. In this context, one may refer [2-20]. More specifically, one may refer “The on – line encyclopedia of integer sequences” (A031396, A130226, A031398) for values of D for which the negative Pell equation  $y^2 = Dx^2 - 1$  is solvable or not. In this communication, the negative Pell equation given by  $y^2 = 3x^2 - 2$  is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

**METHOD OF ANALYSIS:**

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 3x^2 - 2 \tag{1}$$

whose smallest positive integer solution is  $x_0 = 1, y_0 = 1$ .

To obtain the other solutions of (1), consider the Pell equation  $y^2 = 3x^2 + 1$ , whose solution is given by,

$$\tilde{y}_n = \frac{1}{2}f_n, \quad \tilde{x}_n = \frac{1}{2\sqrt{3}}g_n$$

where,

$$f_n = (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1}$$

$$g_n = (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}$$

Applying Brahamagupta Lemma between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$ , the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n$$

$$y_{n+1} = \frac{1}{2}f_n + \frac{\sqrt{3}}{2}g_n$$

The recurrence relations satisfied by the solutions  $x$  and  $y$  are given by,

$$x_{n+1} - 4x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 4y_{n+2} + y_{n+3} = 0$$

Some numerical examples of  $x$  &  $y$  satisfying (1) are given in the table (1) below.

**Table 1: Examples**

$n$	$x_n$	$y_n$
0	1	1
1	3	5
2	11	19
3	41	71
4	153	265

From the above table, we observe some interesting relations among the solutions which are presented below.

- $x_n$  and  $y_n$  both are always odd.
- Each of the following expressions is a nasty number:

$$\diamond 30x_{2n+2} - 6x_{2n+3} + 12$$

$$\diamond 27x_{2n+2} - 3y_{2n+3} + 12$$

$$\diamond 114x_{2n+3} - 30x_{2n+4} + 12$$

$$\diamond 9x_{2n+3} - 15y_{2n+2} + 12$$

$$\diamond 54x_{2n+3} - 30y_{2n+3} + 12$$

$$\diamond 99x_{2n+3} - 15y_{2n+4} + 12$$

$$\diamond 27x_{2n+4} - 57y_{2n+3} + 12$$

$$\diamond 198x_{2n+4} - 114y_{2n+4} + 12$$

$$\begin{aligned} & \diamond 6y_{2n+3} - 18y_{2n+2} + 12 \\ & \diamond 18y_{2n+4} - 66y_{2n+3} + 12 \\ & \diamond \frac{(57x_{2n+2} - 3x_{2n+4} + 24)}{2} \\ & \diamond \frac{(198x_{2n+2} - 6y_{2n+4} + 84)}{7} \\ & \diamond \frac{(18x_{2n+4} - 114y_{2n+2} + 84)}{7} \\ & \diamond \frac{(3y_{2n+4} - 33y_{2n+2} + 24)}{2} \end{aligned}$$

3. Each of the following expressions is a cubical integer:

$$\begin{aligned} & \diamond (5x_{3n+3} - x_{3n+4}) + 3(5x_{n+1} - x_{n+2}) \\ & \diamond 48(19x_{n+1} - x_{n+3}) + 16(19x_{3n+3} - x_{3n+5}) \\ & \diamond 12(9x_{n+1} - y_{n+2}) + 4(9x_{3n+3} - y_{3n+4}) \\ & \diamond 147(33x_{n+1} - y_{n+3}) + 49(33x_{3n+3} - y_{3n+5}) \\ & \diamond 3(19x_{n+2} - 5x_{n+3}) + (19x_{3n+4} - 5x_{3n+5}) \\ & \diamond 12(3x_{n+2} - 5y_{n+1}) + 4(3x_{3n+4} - 5y_{3n+3}) \\ & \diamond 3(9x_{n+2} - 5y_{n+2}) + (9x_{3n+4} - 5y_{3n+4}) \\ & \diamond 12(33x_{n+2} - 5y_{n+3}) + 4(33x_{3n+4} - 5y_{3n+5}) \\ & \diamond 147(3x_{n+3} - 19y_{n+1}) + 49(3x_{3n+5} - 19y_{3n+3}) \\ & \diamond 12(9x_{n+3} - 19y_{n+2}) + 4(9x_{3n+5} - 19y_{3n+4}) \\ & \diamond 3(33x_{n+3} - 19y_{n+3}) + (33x_{3n+5} - 19y_{3n+5}) \\ & \diamond 3(y_{n+2} - 3y_{n+1}) + (y_{3n+4} - 3y_{3n+3}) \\ & \diamond 48(y_{n+3} - 11y_{n+1}) + 16(y_{3n+5} - 11y_{3n+3}) \\ & \diamond 3(3y_{n+3} - 11y_{n+2}) + (3y_{3n+5} - 11y_{3n+4}) \end{aligned}$$

4. Relations among the solutions:

- ❖  $y_{n+1} = x_{n+2} - 2x_{n+1}$
- ❖  $y_{n+2} = 2x_{n+2} - x_{n+1}$
- ❖  $y_{n+3} = 7x_{n+2} - 2x_{n+1}$
- ❖  $4y_{n+1} = x_{n+3} - 7x_{n+1}$
- ❖  $2y_{n+2} = x_{n+3} - x_{n+1}$
- ❖  $4y_{n+3} = 7x_{n+3} - x_{n+1}$
- ❖  $2y_{n+1} = y_{n+2} - 3x_{n+1}$
- ❖  $2y_{n+3} = 3x_{n+1} - 7y_{n+2}$
- ❖  $7y_{n+1} = y_{n+3} - 12x_{n+1}$
- ❖  $y_{n+1} = 2x_{n+3} - 7x_{n+2}$
- ❖  $y_{n+2} = x_{n+3} - 2x_{n+2}$
- ❖  $y_{n+3} = 2x_{n+3} - x_{n+2}$
- ❖  $2y_{n+2} = 3x_{n+2} + y_{n+1}$
- ❖  $y_{n+3} = 6x_{n+2} + y_{n+1}$
- ❖  $y_{n+3} = 3x_{n+2} + 2y_{n+2}$
- ❖  $7y_{n+2} = 3x_{n+3} + 2y_{n+1}$
- ❖  $7y_{n+3} = 12x_{n+3} + y_{n+1}$
- ❖  $2y_{n+3} = 3x_{n+3} + y_{n+2}$

**REMARKABLE OBSERVATIONS:**

- I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below.

**Table 2: Hyperbolas**

S. No.	( X , Y )	Hyperbolas
1	$(x_{n+2} - 3x_{n+1}, 5x_{n+1} - x_{n+2})$	$Y^2 - 3X^2 = 4$

2	$(x_{n+3} - 11x_{n+1}, 19x_{n+1} - x_{n+3})$	$Y^2 - 3X^2 = 64$
3	$(y_{n+2} - 5x_{n+1}, 9x_{n+1} - y_{n+2})$	$Y^2 - 3X^2 = 16$
4	$(y_{n+3} - 19x_{n+1}, 33x_{n+1} - y_{n+3})$	$Y^2 - 3X^2 = 196$
5	$(3x_{n+3} - 11x_{n+2}, 19x_{n+2} - 5x_{n+3})$	$Y^2 - 3X^2 = 4$
6	$(3y_{n+1} - x_{n+2}, 3x_{n+2} - 5y_{n+1})$	$Y^2 - 3X^2 = 16$
7	$(3y_{n+2} - 5x_{n+2}, 9x_{n+2} - 5y_{n+2})$	$Y^2 - 3X^2 = 4$
8	$(3y_{n+3} - 19x_{n+2}, 33x_{n+2} - 5y_{n+3})$	$Y^2 - 3X^2 = 16$
9	$(11y_{n+1} - x_{n+3}, 3x_{n+3} - 19y_{n+1})$	$Y^2 - 3X^2 = 196$
10	$(11y_{n+2} - 5x_{n+3}, 9x_{n+3} - 19y_{n+2})$	$Y^2 - 3X^2 = 16$
11	$(11y_{n+3} - 19x_{n+3}, 33x_{n+3} - 19y_{n+3})$	$Y^2 - 3X^2 = 4$
12	$(5y_{n+1} - y_{n+2}, y_{n+2} - 3y_{n+1})$	$3Y^2 - X^2 = 12$
13	$(19y_{n+1} - y_{n+3}, y_{n+3} - 11y_{n+1})$	$3Y^2 - X^2 = 192$
14	$(19y_{n+2} - 5y_{n+3}, 3y_{n+3} - 11y_{n+2})$	$3Y^2 - X^2 = 12$

- II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below.

**Table 3: Parabolas**

S.No.	( X, Y )	Parabolas
1	$(x_{n+2} - 3x_{n+1}, 5x_{2n+2} - x_{2n+3} + 2)$	$3X^2 = Y - 4$
2	$(x_{n+3} - 11x_{n+1}, 19x_{2n+2} - x_{2n+4} + 8)$	$3X^2 = 4Y - 64$
3	$(y_{n+2} - 5x_{n+1}, 9x_{2n+2} - y_{2n+3} + 4)$	$3X^2 = 2Y - 16$
4	$(y_{n+3} - 19x_{n+1}, 33x_{2n+2} - y_{2n+4} + 14)$	$3X^2 = 7Y - 196$
5	$(3x_{n+3} - 11x_{n+2}, 19x_{2n+3} - 5x_{2n+4} + 2)$	$3X^2 = Y - 4$
6	$(3y_{n+1} - x_{n+2}, 3x_{2n+3} - 5y_{2n+2} + 4)$	$3X^2 = 2Y - 16$
7	$(3y_{n+2} - 5x_{n+2}, 9x_{2n+3} - 5y_{2n+3} + 2)$	$3X^2 = Y - 4$
8	$(3y_{n+3} - 19x_{n+2}, 33x_{2n+3} - 5y_{2n+4} + 4)$	$3X^2 = 2Y - 16$
9	$(11y_{n+1} - x_{n+3}, 3x_{2n+4} - 19y_{2n+2} + 14)$	$3X^2 = 7Y - 196$
10	$(11y_{n+2} - 5x_{n+3}, 9x_{2n+4} - 19y_{2n+3} + 4)$	$3X^2 = 2Y - 16$
11	$(11y_{n+3} - 19x_{n+3}, 33x_{2n+4} - 19y_{2n+4} + 2)$	$3X^2 = Y - 4$
12	$(5y_{n+1} - y_{n+2}, y_{2n+3} - 3y_{2n+2} + 2)$	$X^2 = 3Y - 12$
13	$(19y_{n+1} - y_{n+3}, y_{2n+4} - 11y_{2n+2} + 8)$	$X^2 = 12Y - 192$
14	$(19y_{n+2} - 5y_{n+3}, 3y_{2n+4} - 11y_{2n+3} + 2)$	$X^2 = 3Y - 12$

III. Consider  $p = x + y$ ,  $q = x$ . Observe that  $p > q > 0$ . Treat  $p, q$  as the generators of the Pythagorean triangle  $T(X, Y, Z)$ ,

$$\text{where } X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2.$$

Let  $A$  and  $P$  be denote the area and perimeter of the Pythagorean triangle.

Then the following interesting relations are observed:

a)  $2X - 3Y + Z = 4$

b)  $2Z - 5X + \frac{12A}{P} = -4$

c)  $xy = \frac{2A}{P}$

### CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation  $y^2 = 3x^2 - 2$ . As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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